

# PERIODIC HOMOGENIZATION OF STRONGLY NONLINEAR REACTION-DIFFUSION EQUATIONS WITH LARGE REACTION TERMS

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ABSTRACT. We study in this paper the periodic homogenization problem related to a strongly nonlinear reaction-diffusion equation. Owing to the large reaction term, the homogenized equation has a rather quite different form which puts together both the reaction and convection effects. We show in a special case that, the homogenized equation is exactly of a convection-diffusion type. The study relies on a suitable version of the well-known two-scale convergence method.

## 1. INTRODUCTION

The aim of this work is the study of the asymptotic behavior of the solutions of an initial boundary value problem for a strongly nonlinear reaction-diffusion equation with a large reaction term, in a cylinder. The equation reads as:

$$\begin{aligned} \rho\left(\frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial t} &= \operatorname{div} a\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}, Du_\varepsilon\right) + \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}, u_\varepsilon\right) \quad \text{in } Q_T \\ u_\varepsilon &= 0 \quad \text{on } \partial Q \times (0, T) \\ u_\varepsilon(x, 0) &= u^0(x) \in L^2(Q) \end{aligned} \quad (1.1)$$

where  $Q_T = Q \times (0, T)$  is our cylinder and  $k$  is a given positive parameter. The motivation of this study comes essentially from the applicability of the preceding model. In fact, when the function  $a(x, t, y, \tau, \lambda)$  is linear with respect to  $\lambda$ , that is,  $a(x, t, y, \tau, \lambda) = b(x, t, y, \tau) \cdot \lambda$ , the unknown  $u_\varepsilon$  may be viewed as the concentration of some chemical products diffusing in a porous medium of porosity  $\rho(y)$ , with varying diffusivity  $b(x, t, y, \tau)$  and reacting with the background medium by absorption/desorption through the term  $g(y, \tau, r)$  [2]. The fact that the diffusivity depends on the macroscopic variables  $(x, t)$  means that the concentration varies locally (and not uniformly) in the medium. When the diffusivity is nonlinear as it is the case in (1.1) and has the specific form  $a(x, t, y, \tau, \lambda) = b(x, t, y, \tau) |\lambda|^{p-2} \lambda$ , we obtain some model equations of porous media [5] (see also [6]); here  $u_\varepsilon$  is the density of the fluid,  $\rho(y)$  and  $b(x, t, y, \tau)$  are respectively the porosity and the permeability of the medium.

To proceed with the study of our model, we apply general ideas of homogenization [7, 9] and specifically the framework of two-scale convergence introduced in [11] and developed in [1]. Although the homogenization process is standard, it has still some difficulties in our situation. In fact, the diffusion term is nonlinear, and

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the lower order term  $\frac{1}{\varepsilon}g(x/\varepsilon, t/\varepsilon^k, u_\varepsilon)$  is large because of the presence of the factor  $1/\varepsilon$ . To avoid obtaining a resulting homogenized equation of stochastic's type, we assume a centering type condition on the function  $g$ , that is the periodic function  $(y, \tau) \mapsto g(y, \tau, r)$  has zero mean value with respect to the variable  $y$ , which then allows us to express  $g$  as the gradient of a regular function. We also use this condition in both the a priori estimates and the passage to the limit. This produces a limit problem of a completely different type, which puts together both the reaction and convection effects; see Proposition 4. To be more precise, here is the main result of the paper (the assumptions are to be specified later).

**Theorem 1.** *Let  $2 \leq p < \infty$ . Assume hypotheses **A1-A5** hold. For each  $\varepsilon > 0$  let  $u_\varepsilon$  be the unique solution to (1.1). Then there exists a subsequence of  $\varepsilon$  not relabeled such that  $u_\varepsilon \rightarrow u_0$  in  $L^2(Q_T)$  where  $u_0 \in L^p(0, T; W_0^{1,p}(Q))$  is solution to the following boundary value problem:*

$$\begin{cases} \frac{\partial u_0}{\partial t} = \operatorname{div} q(\cdot, \cdot, u_0, Du_0) + q_0(\cdot, \cdot, u_0, Du_0) & \text{in } Q_T \\ u_0 = 0 & \text{on } \partial Q \times (0, T) \\ u_0(x, 0) = u^0(x) & \text{in } Q. \end{cases} \quad (1.2)$$

The main issue in getting (1.2) lies at the level that the derivative with respect to time  $\partial u_\varepsilon / \partial t$  involves a weight function represented by  $\rho(x/\varepsilon)$ . Indeed, with the presence of  $\rho(x/\varepsilon)$  the usual Aubin-Lions compactness result [10, Chap. 1, p. 58] does not apply to our situation, and we use an appropriate one due to Amar et al. [4, Theorem 2.3] and generalizing the former. Also, because of  $\rho(x/\varepsilon)$ , the space of test functions in the homogenization process is strongly modified. In the framework of the usual two-scale convergence, the test functions are usually taken in a space of the type  $\mathcal{C}_{\text{per}}^\infty(Y)$ . Here, because of the function  $\rho$ , this space is reduced to those functions  $u$  in  $\mathcal{C}_{\text{per}}^\infty(Y)$  satisfying the additional *normalized condition*  $\int_Y \rho(y)u(y)dy = 0$ . This condition plays a crucial role in the choice of the correction term  $u_1$ , which must then satisfy the same assumption itself. Another consequence of this choice is that one must prove the density of the space  $\{u \in \mathcal{C}_{\text{per}}^\infty(Y) : \int_Y \rho(y)u(y)dy = 0\}$  in the space  $\{u \in W_{\text{per}}^{1,p}(Y) : \int_Y \rho(y)u(y)dy = 0\}$ . Also, due to the form of the homogenized problem (which might be degenerate) there is no general uniqueness result for the homogenized equation (1.2). However, we show that in some cases, there is uniqueness of the solution to the said problem.

There is a variety of papers dealing with homogenization of operators of the same type as (1.1) but with a linear diffusion term which is not depending on the macroscopic variables  $(x, t)$ . Without any pretension of exhaustiveness we refer to [2] (for the case when  $k = 2$ ), to [15] (in which  $\rho \equiv 1$  and  $k = 2$ , but the behavior in the microscopic time variable being with respect to some ergodic diffusion process  $\xi_{t/\varepsilon^2}$ ) and to [8] in which the following operator is considered:

$$\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div} \left( a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k} \right) Du_\varepsilon \right) + \frac{1}{\varepsilon^{\max(1, k/2)}} g \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}, u_\varepsilon \right) + h \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}, u_\varepsilon \right)$$

with the same assumptions as in [15].

The paper is organized as follows. In Section 2, we recall the concept of two-scale convergence. We adapt it to the situation of the problem (1.1). Section 3 deals with a priori estimates of the solution of the problem (1.1). In Section 4, we give some preliminary results that will be used in the next section. Finally,

Section 5 deals with the homogenization results for (1.1). We also study there a particular case when the homogenized problem possesses a unique solution, and show that the whole sequence converges in that case to the solution of a problem of convection-diffusion type.

We end this section with some notations. All functions are assumed real valued and all function spaces are considered over  $\mathbb{R}$ . Let  $Y = (0, 1)^N$  and let  $F(\mathbb{R}^N)$  be a given function space. We denote by  $F_{\text{per}}(Y)$  the space of functions in  $F_{\text{loc}}(\mathbb{R}^N)$  (when it makes sense) that are  $Y$ -periodic. Given a  $Y$ -periodic function  $\rho$ , we denote by  $F_{\# \rho}(Y)$  the subspace of  $F_{\text{per}}(Y)$  consisting of functions  $u$  for which  $\rho u$  has mean value zero:  $\int_Y \rho(y)u(y)dy = 0$ . As special cases,  $\mathcal{D}_{\text{per}}(Y)$  denotes the space  $\mathcal{C}_{\text{per}}^\infty(Y)$  while  $\mathcal{D}_{\# \rho}(Y)$  stands for the space of those functions  $u$  in  $\mathcal{D}_{\text{per}}(Y)$  for which  $\rho u$  has mean value zero.  $\mathcal{D}'_{\text{per}}(Y)$  stands for the topological dual of  $\mathcal{D}_{\text{per}}(Y)$  which can be identified to the space of periodic distributions in  $\mathcal{D}'(\mathbb{R}^N)$ .

## 2. TWO-SCALE CONVERGENCE

We recall the notion of two-scale convergence [1, 11]. We adapt it to our framework and get the following

**Definition 1.** A sequence  $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q_T)$  ( $1 \leq p < \infty$ ) is said to two-scale converge towards  $u_0 \in L^p(Q_T \times Y \times Z)$  ( $Z = (0, 1)$ ) if, as  $\varepsilon \rightarrow 0$ ,

$$\int_{Q_T} u_\varepsilon(x, t) f\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}\right) dx dt \rightarrow \iint_{Q_T \times Y \times Z} u_0(x, t, y, \tau) f(x, t, y, \tau) dx dt dy d\tau \quad (2.1)$$

for all  $f \in L^{p'}(Q_T; \mathcal{C}_{\text{per}}(Y \times Z))$ . We denote it by  $u_\varepsilon \rightarrow u_0$  in  $L^p(Q_T)$ -2s.

The following two compactness results are well-known in the literature; see e.g. [13] for the exact situation considered here.

**Theorem 2.** Let  $1 < p < \infty$ . Then any bounded sequence in  $L^p(Q_T)$  admits a two-scale convergent subsequence.

**Theorem 3.** Let  $1 < p < \infty$ . Let  $(u_\varepsilon)_{\varepsilon \in E}$  (where  $E$  is an ordinary sequence of real numbers converging to zero with  $\varepsilon$ ) be a bounded sequence in  $L^p(0, T; W_0^{1,p}(Q))$ . There exist a subsequence of  $E$  denoted by  $E'$ , and a couple  $(u_0, u_1) \in L^p(0, T; W_0^{1,p}(Q)) \times L^p(Q_T \times Z; W_{\text{per}}^{1,p}(Y))$  such that, as  $E' \ni \varepsilon \rightarrow 0$ ,

$$u_\varepsilon \rightarrow u_0 \text{ in } L^p(0, T; W_0^{1,p}(Q))\text{-weak}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_j} \rightarrow \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^p(Q_T)\text{-2s } (1 \leq j \leq N).$$

In Theorem 3 the function  $u_1$  is unique up to an additive function of variables  $x, t, \tau$ . We need to fix its choice in accordance with the needs in the sequel. For that, let us recall the definition of the space  $W_{\# \rho}^{1,p}(Y)$  for a given positive function  $\rho \in L^\infty_{\text{per}}(Y)$  with non zero mean value:

$$W_{\# \rho}^{1,p}(Y) = \left\{ u \in W_{\text{per}}^{1,p}(Y) : \int_Y \rho(y)u(y)dy = 0 \right\}.$$

$W_{\# \rho}^{1,p}(Y)$  is a closed subspace of  $W_{\text{per}}^{1,p}(Y)$  since it is the kernel of the continuous linear functional  $u \mapsto \int_Y \rho(y)u(y)dy$  defined on  $W_{\text{per}}^{1,p}(Y)$ . The following version of Theorem 3 will be used in the sequel.

**Theorem 4.** *Assumptions are those of Theorem 3. Assume moreover that  $p \geq 2$  and that there exists a function  $u_0 \in L^p(0, T; W_0^{1,p}(Q))$  such that  $u_\varepsilon \rightarrow u_0$  in  $L^2(Q_T)$  as  $\varepsilon \rightarrow 0$ . Then there exists a subsequence  $E'$  of  $E$  and a function  $u_1 \in L^p(Q_T \times Z; W_{\# \rho}^{1,p}(Y))$  such that, as  $E' \ni \varepsilon \rightarrow 0$ ,*

$$\frac{\partial u_\varepsilon}{\partial x_j} \rightarrow \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^p(Q_T)\text{-}2s \text{ (} 1 \leq j \leq N \text{)}. \quad (2.2)$$

*Proof.* Let  $u_1^\# \in L^p(Q_T \times Z; W_{\text{per}}^{1,p}(Y))$  be such that Theorem 3 holds with  $u_1^\#$  in place of  $u_1$  in that theorem. Set

$$u_1(x, t, y, \tau) = u_1^\#(x, t, y, \tau) - \frac{1}{\int_Y \rho(y) dy} \int_Y \rho(y) u_1^\#(x, t, y, \tau) dy$$

for  $(x, t, y, \tau) \in Q_T \times Y \times Z$ . Then  $u_1 \in L^p(Q_T \times Z; W_{\# \rho}^{1,p}(Y))$  and moreover  $\partial u_1 / \partial y_i = \partial u_1^\# / \partial y_i$  ( $1 \leq i \leq N$ ), so that (2.2) holds.  $\square$

**Remark 1.** In case  $\rho \equiv 1$ , we retrieve the result of [13] since in that case  $W_{\# \rho}^{1,p}(Y) = W_{\#}^{1,p}(Y) := \{u \in W_{\text{per}}^{1,p}(Y) : \int_Y u dy = 0\}$ . Throughout the rest of the paper, we assume without loss of generality that  $\int_Y \rho dy = 1$ .

### 3. STATEMENT OF THE PROBLEM: A PRIORI ESTIMATES AND COMPACTNESS RESULT FOR THE SOLUTION

**3.1. Problem setting.** Let  $Q$  be a bounded Lipschitz domain of  $\mathbb{R}^N$  and  $T$  a positive real number. By  $Q_T$  we denote the cylinder  $Q \times (0, T)$ . Our aim is to study the asymptotic behavior of the sequence of solutions to (1.1). We begin this section by setting the necessary conditions under which such a study can be made possible. For instance, we assume that the coefficients of (1.1) are constrained as follows:

**A1** The function  $a : (x, t, y, \tau, \lambda) \mapsto a(x, t, y, \tau, \lambda)$  from  $\overline{Q}_T \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$  into  $\mathbb{R}^N$  satisfies the properties that:

For each fixed  $(x, t) \in \overline{Q}_T$  and  $\lambda \in \mathbb{R}^N$ ,  $a(x, t, \cdot, \cdot, \lambda)$  is measurable (3.1)

$a(x, t, y, \tau, 0) = 0$  almost everywhere (a.e.) in  $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$   
and for all  $(x, t) \in \overline{Q}_T$ . (3.2)

There are three constants  $c_0, c_1, c_2 > 0$  and a continuity modulus  $\omega$  (i.e., a nondecreasing continuous function on  $[0, +\infty)$  such that  $\omega(0) = 0, \omega(r) > 0$  if  $r > 0$ , and  $\omega(r) = 1$  if  $r > 1$ ) such that a.e. in  $(y, \tau) \in \mathbb{R}^N \times \mathbb{R}$ ,

$$\begin{aligned} & \text{(i)} \quad (a(x, t, y, \tau, \lambda) - a(x, t, y, \tau, \lambda')) \cdot (\lambda - \lambda') \geq c_1 |\lambda - \lambda'|^p \\ & \text{(ii)} \quad |a(x, t, y, \tau, \lambda)| \leq c_2 (1 + |\lambda|^{p-1}) \\ & \text{(iii)} \quad |a(x, t, y, \tau, \lambda) - a(x', t', y, \tau, \lambda')| \\ & \quad \leq \omega(|x - x'| + |t - t'|) (1 + |\lambda|^{p-1} + |\lambda'|^{p-1}) + c_0 (1 + |\lambda| + |\lambda'|)^{p-2} |\lambda - \lambda'| \\ & \text{for all } (x, t), (x', t') \in \overline{Q}_T \text{ and all } \lambda, \lambda' \in \mathbb{R}^N, \text{ where the dot} \\ & \text{denotes the usual Euclidean inner product in } \mathbb{R}^N \text{ and } |\cdot| \text{ the associated} \\ & \text{norm.} \end{aligned} \quad (3.3)$$

**A2 Lipschitz continuity.** The function  $g$  is continuous on  $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$  and there is  $C > 0$  such that for any  $(y, \tau) \in \mathbb{R}^{N+1}$  and  $r, r_1, r_2 \in \mathbb{R}$

$$\begin{aligned} |\partial_r g(y, \tau, r)| &\leq C \\ |\partial_r g(y, \tau, r_1) - \partial_r g(y, \tau, r_2)| &\leq C |r_1 - r_2| (1 + |r_1| + |r_2|)^{-1}. \end{aligned}$$

**A3 Equilibrium condition.** We assume that 0 is a possible equilibrium solution of (1.1), that is,  $g(y, \tau, 0) = 0$  for any  $(y, \tau) \in \mathbb{R}^{N+1}$ .

**A4 Positivity.** The density function  $\rho \in L^\infty(\mathbb{R}^N)$  and there exists  $\Lambda > 0$  such that

$$\Lambda^{-1} \leq \rho(y) \leq \Lambda \text{ for a.e. } y \in \mathbb{R}^N.$$

We also assume without loss of generality that

$$\int_Y \rho(y) dy = 1.$$

**A5 Periodicity hypothesis.** The density function  $\rho$  is  $Y$ -periodic, the function  $(y, \tau) \mapsto a(x, t, y, \tau, \lambda)$  is  $Y \times Z$ -periodic for any fixed  $x, t, \lambda$ . We assume also that  $g(\cdot, \cdot, r) \in \mathcal{C}_{\text{per}}(Y \times Z)$  for any  $r \in \mathbb{R}$  with  $\int_Y g(y, \tau, r) dy = 0$  for all  $(\tau, r) \in \mathbb{R}^2$ . We easily infer from the Fredholm alternative the existence of a unique  $R(\cdot, \cdot, r) \in \mathcal{C}_{\text{per}}(Y \times Z)$  such that  $\Delta_y R(\cdot, \cdot, r) = g(\cdot, \cdot, r)$  and  $\int_Y R(\cdot, \tau, r) dy = 0$  for all  $\tau, r \in \mathbb{R}$ , where  $\Delta_y$  stands for the Laplacian with respect to the variable  $y$ . Moreover  $R(\cdot, \cdot, r)$  is at least twice differentiable with respect to  $y$ . Let  $G(y, \tau, r) = D_y R(y, \tau, r)$ . Thanks to **A2** and **A3** we see that

$$|G(y, \tau, r)| \leq C |r|, \quad |\partial_r G(y, \tau, r)| \leq C, \quad (3.4)$$

$$|\partial_r G(y, \tau, r_1) - \partial_r G(y, \tau, r_2)| \leq C |r_1 - r_2| (1 + |r_1| + |r_2|)^{-1} \quad (3.5)$$

where  $\partial_r G$  denotes the partial derivative of  $G$  with respect to  $r$ .

As regards the definition of the trace functions  $(x, t) \mapsto a(x, t, x/\varepsilon, t/\varepsilon^k, Du_\varepsilon(x, t))$ ,  $(x, t) \mapsto g(x/\varepsilon, t/\varepsilon^k, u_\varepsilon(x, t))$  and  $x \mapsto \rho(x/\varepsilon)$  here denoted respectively by  $a^\varepsilon(\cdot, Du_\varepsilon)$ ,  $g^\varepsilon(u_\varepsilon)$  and  $\rho^\varepsilon$ , this has been extensively discussed in many papers (see e.g. [12, 20]). These functions are well-defined and satisfy properties of the same type as in **A1-A4**. Due to both the positivity assumption on the density function  $\rho$  and the Lipschitzity of the function  $g(y, \tau, \cdot)$ , one can show in a standard fashion that the problem (1.1) admits a unique solution  $u_\varepsilon$ , which moreover belongs to the space  $L^p(0, T; W_0^{1,p}(Q)) \cap \mathcal{C}(0, T; L^2(Q))$ ; see e.g., [3, 16].

**3.2. A priori estimates and compactness.** We will denote by  $(\cdot, \cdot)$  the duality pairing between  $W_0^{1,p}(Q)$  and its topological dual  $W^{-1,p'}(Q)$ . The symbols  $|\cdot|_{L^p}$  and  $\|\cdot\|$  will stand for the respective norms of  $L^p(Q)$  and  $W_0^{1,p}(Q)$ . Throughout  $C$  will denote a generic constant independent of  $\varepsilon$ . The following uniform a priori estimates hold.

**Lemma 1.** *Under assumptions **A1-A5** the following estimates hold true for  $2 \leq p < \infty$ :*

$$\sup_{0 \leq t \leq T} |u_\varepsilon(t)|_{L^2}^2 \leq C, \quad (3.6)$$

$$\int_0^T \|u_\varepsilon(t)\|^p dt \leq C \quad (3.7)$$

where  $C$  is a positive constant which does not depend on  $\varepsilon$ .

*Proof.* We have  $u_\varepsilon \in L^p(0, T; W_0^{1,p}(Q)) \cap \mathcal{C}(0, T; L^2(Q))$  and the following energy equation holds:

$$\begin{aligned} & \left| (\rho^\varepsilon)^{\frac{1}{2}} u_\varepsilon(t) \right|_{L^2}^2 - \left| (\rho^\varepsilon)^{\frac{1}{2}} u^0 \right|_{L^2}^2 + 2 \int_0^t \int_Q a^\varepsilon(\cdot, Du_\varepsilon(s)) \cdot Du_\varepsilon(s) dx ds \\ &= 2 \int_0^t \int_Q \frac{1}{\varepsilon} g^\varepsilon(u_\varepsilon(s)) u_\varepsilon(s) dx ds. \end{aligned} \quad (3.8)$$

But using the representation  $G(y, \tau, r) = D_y R(y, \tau, r)$  obtained above, we get

$$\frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}, u_\varepsilon\right) = \operatorname{div} G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}, u_\varepsilon\right) - \partial_r G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k}, u_\varepsilon\right) \cdot Du_\varepsilon.$$

Putting the above expression in (3.8) we obtain

$$\begin{aligned} & \left| (\rho^\varepsilon)^{\frac{1}{2}} u_\varepsilon(t) \right|_{L^2}^2 + 2 \int_0^t \int_Q a^\varepsilon(\cdot, Du_\varepsilon) \cdot Du_\varepsilon dx ds \\ &= -2 \int_0^t \int_Q G^\varepsilon(u_\varepsilon) \cdot Du_\varepsilon dx ds - 2 \int_0^t \int_Q (\partial_r G^\varepsilon(u_\varepsilon) \cdot Du_\varepsilon) u_\varepsilon dx ds + \left| (\rho^\varepsilon)^{\frac{1}{2}} u^0 \right|_{L^2}^2, \end{aligned}$$

where  $G^\varepsilon(u_\varepsilon)$  and  $\partial_r G^\varepsilon(u_\varepsilon)$  are defined exactly as  $g^\varepsilon(u_\varepsilon)$ . Thanks to Assumptions **A1-A5** one can easily see that

$$\begin{aligned} \Lambda^{-1} |u_\varepsilon(t)|_{L^2}^2 + 2c_1 \int_0^t \int_Q |Du_\varepsilon|^p dx ds &\leq 2C \int_0^t \int_Q |u_\varepsilon| |Du_\varepsilon| dx ds \\ &\quad + 2C \int_0^t \int_Q |u_\varepsilon| |Du_\varepsilon| dx ds + \Lambda |u^0|_{L^2}^2. \end{aligned}$$

But by Young's inequality, we have, for any positive  $\delta$ ,

$$4C \int_0^t \int_Q |u_\varepsilon| |Du_\varepsilon| dx ds \leq \int_0^t \int_Q \left( \frac{4C\delta^{-p'}}{p'} |u_\varepsilon|^{p'} + \frac{4C\delta^p}{p} |Du_\varepsilon|^p \right) dx ds.$$

Choosing  $\delta$  in such a way that  $\frac{4C\delta^p}{p} = c_1$  we get

$$\Lambda^{-1} |u_\varepsilon(t)|_{L^2}^2 + c_1 \int_0^t \int_Q |Du_\varepsilon|^p dx ds \leq C \int_0^t |u_\varepsilon|_{L^{p'}}^{p'} ds + K$$

where  $K = \Lambda |u^0|_{L^2}^2$ . But as  $p' \leq 2$ , we have  $|u_\varepsilon|_{L^{p'}}^{p'} \leq C |u_\varepsilon|_{L^2}^{p'}$ , thus

$$\Lambda^{-1} |u_\varepsilon(t)|_{L^2}^2 + c_1 \int_0^t \int_Q |Du_\varepsilon|^p dx ds \leq C \int_0^t |u_\varepsilon|_{L^2}^{p'} ds, \quad (3.9)$$

hence

$$\Lambda^{-1} |u_\varepsilon(t)|_{L^2}^2 \leq C \int_0^t |u_\varepsilon|_{L^2}^{p'} ds.$$

But, since  $p' \leq 2$ , there is a positive constant  $k_1$  independent of  $\varepsilon$  such that  $|u_\varepsilon(t)|_{L^2}^{p'} \leq k_1(1 + |u_\varepsilon(t)|_{L^2}^2)$ . Thus

$$|u_\varepsilon(t)|_{L^2}^2 \leq C + C \int_0^t |u_\varepsilon(s)|_{L^2}^2 ds$$

where here,  $C = C(T) > 0$ . By the application of Gronwall inequality we get at once (3.6). We then deduce (3.7) from (3.9).  $\square$

The next result should be of capital interest for the sequel.

**Proposition 1.** *The family  $(u_\varepsilon)_{\varepsilon>0}$  is relatively compact in the space  $L^2(Q_T)$ .*

*Proof.* It follows from Eq. (1.1) that

$$\begin{aligned} \left\| \rho^\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^{p'}(0,T;W^{-1,p'}(Q))}^{p'} &\leq C \int_0^T \|\operatorname{div} a^\varepsilon(\cdot, Du_\varepsilon)\|_{W^{-1,p'}(Q)}^{p'} dt \\ &\quad + C \int_0^T \left\| \frac{1}{\varepsilon} g^\varepsilon(u_\varepsilon) \right\|_{W^{-1,p'}(Q)}^{p'} dt, \end{aligned}$$

and, thanks to (3.7) we easily get

$$\int_0^T \|\operatorname{div} a^\varepsilon(\cdot, Du_\varepsilon)\|_{W^{-1,p'}(Q)}^{p'} dt \leq C. \quad (3.10)$$

Next

$$\left\| \frac{1}{\varepsilon} g^\varepsilon(u_\varepsilon) \right\|_{W^{-1,p'}(Q)} = \sup_{\substack{\phi \in W_0^{1,p}(Q) \\ \|\phi\|=1}} \left| \int_Q G^\varepsilon(u_\varepsilon) \cdot D\phi dx + \int_Q (\partial_r G^\varepsilon(u_\varepsilon) \cdot Du_\varepsilon) \phi dx \right|.$$

By using condition **A5** (see especially (3.4)) associated to the Poincaré's inequality we get that

$$\begin{aligned} \left\| \frac{1}{\varepsilon} g^\varepsilon(u_\varepsilon) \right\|_{W^{-1,p'}(Q)} &\leq \sup_{\phi \in W_0^{1,p}(Q), \|\phi\|=1} (C \|u_\varepsilon(s)\|_{L^2} + C \|u_\varepsilon(s)\| \|\phi\|) \\ &\leq C \|u_\varepsilon(s)\|_{L^2} + C \|u_\varepsilon(s)\|. \end{aligned}$$

It therefore follows from the estimates (3.6)-(3.7) and the Hölder's inequality that

$$\int_0^T \left\| \frac{1}{\varepsilon} g^\varepsilon(u_\varepsilon) \right\|_{W^{-1,p'}(Q)}^{p'} dt \leq C. \quad (3.11)$$

Thus we infer from (3.10)-(3.11) that

$$\left\| \rho^\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^{p'}(0,T;W^{-1,p'}(Q))} \leq C.$$

Hence, setting

$$\mathcal{W}_\varepsilon = \{u \in L^p(0,T;W_0^{1,p}(Q)) : (\rho^\varepsilon u)' \in L^{p'}(0,T;W^{-1,p'}(Q))\}$$

which is a Banach space under the norm

$$\|u\|_{\mathcal{W}_\varepsilon} = \|u\|_{L^p(0,T;W_0^{1,p}(Q))} + \|(\rho^\varepsilon u)'\|_{L^{p'}(0,T;W^{-1,p'}(Q))},$$

we have that

$$\|u_\varepsilon\|_{\mathcal{W}_\varepsilon} \leq C \text{ for any } \varepsilon > 0$$

where  $C$  is independent of  $\varepsilon$ . Since  $\int_Y \rho dy \neq 0$ , we therefore deduce from [4, Theorem 2.3] that  $(u_\varepsilon)_{\varepsilon>0}$  is relatively compact in  $L^2(Q_T)$ .  $\square$

## 4. PRELIMINARY RESULTS

Let  $2 \leq p < \infty$ . The following Gelfand triplet

$$W_{\#\rho}^{1,p}(Y) \subset L_{\#\rho}^2(Y) \subset (W_{\#\rho}^{1,p}(Y))'$$

holds, with continuous embeddings,  $(W_{\#\rho}^{1,p}(Y))'$  being the topological dual of  $W_{\#\rho}^{1,p}(Y)$ ; this can be seen by showing that the space  $W_{\#\rho}^{1,p}(Y)$  is densely embedded in  $L_{\#\rho}^2(Y)$  (this follows by repeating the proof of the forthcoming Lemma 2). It is also a fact that the topological dual of  $L_{\text{per}}^p(Z; W_{\#\rho}^{1,p}(Y))$  is  $L_{\text{per}}^{p'}(Z; [W_{\#\rho}^{1,p}(Y)]')$ ; this can be easily seen from the fact that  $W_{\#\rho}^{1,p}(Y)$  is reflexive and  $L_{\text{per}}^p(Z; W_{\#\rho}^{1,p}(Y))$  is isometrically isomorphic to  $L^p(\mathbb{T}; W_{\#\rho}^{1,p}(Y))$  where  $\mathbb{T}$  is the 1-dimensional torus. We denote by  $(\cdot, \cdot)$  (resp.  $[\cdot, \cdot]$ ) the duality pairing between  $W_{\#\rho}^{1,p}(Y)$  (resp.  $L_{\text{per}}^p(Z; W_{\#\rho}^{1,p}(Y))$ ) and  $[W_{\#\rho}^{1,p}(Y)]'$  (resp.  $L_{\text{per}}^{p'}(Z; [W_{\#\rho}^{1,p}(Y)]')$ ). For the above reasons we have,

$$[u, v] = \int_0^1 (u(\tau), v(\tau)) d\tau$$

for  $u \in L_{\text{per}}^{p'}(Z; [W_{\#\rho}^{1,p}(Y)]')$  and  $v \in L_{\text{per}}^p(Z; W_{\#\rho}^{1,p}(Y))$ , and

$$(u, \varphi) = \int_Y u(y) \varphi(y) dy$$

for all  $u \in L_{\#\rho}^2(Y)$  and  $\varphi \in W_{\#\rho}^{1,p}(Y)$ .

The following important density result will be used throughout the paper.

**Lemma 2.** *The space*

$$\mathcal{D}_{\#\rho}(Y) = \left\{ u \in \mathcal{D}_{\text{per}}(Y) : \int_Y \rho u dy = 0 \right\}$$

*is dense in  $W_{\#\rho}^{1,p}(Y)$ .*

*Proof.* Let  $L$  be a continuous linear functional on  $W_{\#\rho}^{1,p}(Y)$  verifying  $L(v) = 0$  for all  $v \in \mathcal{D}_{\#\rho}(Y)$ . We need to check that  $L(v) = 0$  for all  $v \in W_{\#\rho}^{1,p}(Y)$ . By the Hahn-Banach theorem,  $L$  extends to a (possibly non unique!) continuous linear functional  $\tilde{L}$  on  $W_{\text{per}}^{1,p}(Y)$  and so, there exists  $(u_i)_{0 \leq i \leq N} \subset (L_{\text{per}}^{p'}(Y))^{N+1}$  such that

$$\tilde{L}(v) = \int_Y u_0 v dy + \sum_{i=1}^N \int_Y u_i \frac{\partial v}{\partial y_i} dy \text{ for all } v \in W_{\text{per}}^{1,p}(Y).$$

Let  $v \in \mathcal{D}_{\text{per}}(Y)$ ; since  $v - M_y(\rho v) \in \mathcal{D}_{\#\rho}(Y)$  (where  $M_y(\rho v) = \int_Y \rho v dy$ ), we have  $\tilde{L}(v - M_y(\rho v)) = L(v - M_y(\rho v)) = 0$ , or equivalently,

$$L(v) = M_y(\rho v) \int_Y u_0 dy \text{ for all } v \in \mathcal{D}_{\text{per}}(Y). \quad (4.1)$$

But the linear functional  $v \mapsto M_y(\rho v)$  defined on  $\mathcal{D}_{\text{per}}(Y)$ , is continuous on  $\mathcal{D}_{\text{per}}(Y)$  endowed with the  $W_{\text{per}}^{1,p}(Y)$ -norm, so that (4.1) still holds true for  $v \in W_{\text{per}}^{1,p}(Y)$  (by the density of  $\mathcal{D}_{\text{per}}(Y)$  in  $W_{\text{per}}^{1,p}(Y)$ ). Therefore taking  $v \in W_{\#\rho}^{1,p}(Y)$  we get  $L(v) = 0$ . This completes the proof.  $\square$

The following obvious result will be used in the sequel.



**Lemma 3.** *Let  $u \in \mathcal{D}'_{\text{per}}(Y \times Z)$ . We still write  $u$  for  $u|_{\mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\# \rho}(Y)]}$  (the restriction of  $u$  to  $\mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\# \rho}(Y)]$ ). Assume  $u$  is continuous on  $\mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\# \rho}(Y)]$  under the  $L^p_{\text{per}}(Z; W^{1,p}_{\# \rho}(Y))$ -norm. Then  $u \in L^{p'}_{\text{per}}(Z; [W^{1,p}_{\# \rho}(Y)]')$  and further*

$$\langle u, \varphi \rangle = \int_0^1 (u(\tau), \varphi(\cdot, \tau)) d\tau$$

for all  $\varphi \in \mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\# \rho}(Y)]$ , where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $\mathcal{D}'_{\text{per}}(Y \times Z)$  and  $\mathcal{D}_{\text{per}}(Y \times Z)$ , whereas the right-hand side denotes the product of  $u$  and  $\varphi$  in the duality between  $L^{p'}_{\text{per}}(Z; [W^{1,p}_{\# \rho}(Y)]')$  and  $L^p_{\text{per}}(Z; W^{1,p}_{\# \rho}(Y))$  as stated above.

The next result will prove very efficient in the homogenization process in the case when  $k = 2$ .

**Lemma 4.** *Let  $\psi \in \mathcal{C}_0^\infty(Q_T) \otimes (\mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\# \rho}(Y)])$ . Let  $(u_\varepsilon)_{\varepsilon \in E}$ ,  $E'$  and  $(u_0, u_1)$  be as in Theorem 4. Then, as  $E' \ni \varepsilon \rightarrow 0$*

$$\int_{Q_T} \frac{1}{\varepsilon} u_\varepsilon \rho^\varepsilon \psi^\varepsilon dx dt \rightarrow \int_{Q_T} [\rho u_1(x, t), \psi(x, t)] dx dt$$

where  $\psi^\varepsilon(x, t) = \psi(x, t, x/\varepsilon, t/\varepsilon^k)$  for  $(x, t) \in Q_T$ .

*Proof.* We recall that the space  $\mathcal{D}_{\# \rho}(Y)$  consists of those functions  $\psi$  in  $\mathcal{D}_{\text{per}}(Y)$  with the property  $\int_Y \rho \psi dy = 0$ . With this in mind, let  $\psi$  be as in the statement of the lemma. Since  $\rho \psi$  is in  $\mathcal{C}_0^\infty(Q_T) \otimes (\mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\text{per}}(Y)])$  and verifies  $\int_Y \rho \psi dy = 0$ , the result follows at once by the application of [13, Lemma 3.4].  $\square$

Let  $\mathcal{R} : L^2_{\text{per}}(Y) \rightarrow L^2_{\text{per}}(Y)$  be defined by  $\mathcal{R}u = \rho u$ . Then  $\mathcal{R}$  is a non-negative and bounded linear self-adjoint operator. By the positivity of  $\rho$ , its kernel is reduced to 0. We denote by  $L^2_\rho(Y)$  the completion of  $L^2_{\text{per}}(Y)$  with respect to the norm  $\|u\|_+ = \|\rho^{1/2}u\|_{L^2(Y)}$ .

Now, for  $u \in L^2_{\text{per}}(Z; L^2_{\text{per}}(Y))$  we define  $\mathcal{R}u$  as follows:

$$\mathcal{R}u(\tau) = \mathcal{R}(u(\tau)) \text{ for a.e. } \tau \in Z = (0, 1),$$

and we get an operator  $\mathcal{R} : L^2_{\text{per}}(Z; L^2_{\text{per}}(Y)) \rightarrow L^2_{\text{per}}(Z; L^2_{\text{per}}(Y))$ . Finally let  $\mathcal{V} = L^p_{\text{per}}(Z; W^{1,p}_{\# \rho}(Y))$  and its topological dual  $\mathcal{V}' = L^{p'}_{\text{per}}(Z; [W^{1,p}_{\# \rho}(Y)]')$ . Viewing  $(\mathcal{R})' \equiv \rho \partial / \partial \tau$  as an unbounded operator defined from  $\mathcal{V}$  into  $\mathcal{V}'$ , its domain is

$$\mathcal{W} = \left\{ v \in \mathcal{V} : \rho \frac{\partial v}{\partial \tau} \in \mathcal{V}' \right\}.$$

With the norm  $\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|\rho \frac{\partial v}{\partial \tau}\|_{\mathcal{V}'}$ ,  $\mathcal{W}$  is a Banach space and the following result holds.

**Proposition 2.** *The space  $\mathcal{W}$  is continuously embedded into  $\mathcal{C}([0, 1]; L^2_\rho(Y))$ , that is, there is a positive constant  $c$  such that*

$$\sup_{0 \leq \tau \leq 1} \left\| \rho^{\frac{1}{2}} u(\tau) \right\|_{L^2(Y)} \leq c \|u\|_{\mathcal{W}}$$

for all  $u \in \mathcal{W}$ . Moreover

$$\left[ \rho \frac{\partial u}{\partial \tau}, v \right] = - \left[ \rho \frac{\partial v}{\partial \tau}, u \right] \quad (4.2)$$

for all  $u, v \in \mathcal{W}$ .

*Proof.* The fact that  $\mathcal{W}$  embeds continuously into  $\mathcal{C}([0, 1]; L^2_\rho(Y))$  follows from [14]; see also [16, Proposition 4.1]. Still from the same references, we have that, for  $u, v \in \mathcal{W}$ ,

$$\left[ \rho \frac{\partial u}{\partial \tau}, v \right] + \left[ \rho \frac{\partial v}{\partial \tau}, u \right] = \int_Y \rho u(1) v(1) dy - \int_Y \rho u(0) v(0) dy,$$

and by the  $Z$ -periodicity of  $u$  and  $v$ , it follows that the right-hand side of the above equality is zero, hence (4.2).  $\square$

By repeating the proof of Lemma 2 one can show that the space  $\mathcal{E} = \mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\# \rho}(Y)]$  is dense in  $\mathcal{W}$ . The operator  $\rho \partial / \partial \tau$  will be useful in the homogenization process for the case  $k = 2$ . This being so, the Lemma 4 has a crucial corollary.

**Corollary 1.** *Let the hypotheses be those of Lemma 4. Assume moreover that  $u_1 \in \mathcal{W}$  and that  $k = 2$ . Then, as  $E' \ni \varepsilon \rightarrow 0$ ,*

$$\int_{Q_T} \varepsilon u_\varepsilon \rho^\varepsilon \frac{\partial \psi^\varepsilon}{\partial t} dx dt \rightarrow - \int_{Q_T} \left[ \rho \frac{\partial u_1}{\partial \tau}(x, t), \psi(x, t) \right] dx dt.$$

*Proof.* We have

$$\int_{Q_T} \varepsilon u_\varepsilon \rho^\varepsilon \frac{\partial \psi^\varepsilon}{\partial t} dx dt = \varepsilon \int_{Q_T} u_\varepsilon \rho^\varepsilon \left( \frac{\partial \psi}{\partial t} \right)^\varepsilon dx dt + \frac{1}{\varepsilon} \int_{Q_T} u_\varepsilon \rho^\varepsilon \left( \frac{\partial \psi}{\partial \tau} \right)^\varepsilon dx dt.$$

Since  $\frac{\partial \psi}{\partial \tau}$  is in  $\mathcal{C}_0^\infty(Q_T) \otimes (\mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\# \rho}(Y)])$ , we infer from Lemma 4 that, as  $E' \ni \varepsilon \rightarrow 0$ ,

$$\int_{Q_T} \varepsilon u_\varepsilon \rho^\varepsilon \frac{\partial \psi^\varepsilon}{\partial t} dx dt \rightarrow \int_{Q_T} \left[ \int_Z \left( \rho u_1(x, t, \cdot, \tau), \frac{\partial \psi}{\partial \tau}(x, t, \cdot, \tau) \right) d\tau \right] dx dt.$$

But

$$\begin{aligned} \int_Z \left( \rho u_1(x, t, \cdot, \tau), \frac{\partial \psi}{\partial \tau}(x, t, \cdot, \tau) \right) d\tau &= \left[ \rho u_1(x, t, \cdot, \cdot), \frac{\partial \psi}{\partial \tau}(x, t, \cdot, \cdot) \right] \\ &= - \left[ \rho \frac{\partial u_1}{\partial \tau}(x, t, \cdot, \cdot), \psi(x, t, \cdot, \cdot) \right], \end{aligned}$$

where in the last equality, we have used (4.2) (see Proposition 2).  $\square$

We will also need the following

**Lemma 5.** *Let  $g : \mathbb{R}_y^N \times \mathbb{R}_\tau \times \mathbb{R}_r \rightarrow \mathbb{R}$  be a function verifying the following conditions:*

- (i)  $|\partial_r g(y, \tau, u)| \leq C$
- (ii)  $g(\cdot, \cdot, r) \in \mathcal{C}_{\text{per}}(Y \times Z)$ .

*Let  $(u_\varepsilon)_\varepsilon$  be a sequence in  $L^2(Q_T)$  such that  $u_\varepsilon \rightarrow u_0$  in  $L^2(Q_T)$  as  $\varepsilon \rightarrow 0$ , where  $u_0 \in L^2(Q_T)$ . Then, setting  $g^\varepsilon(u_\varepsilon)(x, t) = g(x/\varepsilon, t/\varepsilon^k, u_\varepsilon(x, t))$  we have, as  $\varepsilon \rightarrow 0$ ,*

$$g^\varepsilon(u_\varepsilon) \rightarrow g(\cdot, \cdot, u_0) \text{ in } L^2(Q_T)\text{-}2s.$$

*Proof.* Assumption (i) implies the Lipschitz condition

$$|g(y, \tau, r_1) - g(y, \tau, r_2)| \leq C |r_1 - r_2| \text{ for all } y, \tau, r_1, r_2. \quad (4.3)$$

Next, observe that from (ii) and (4.3), the function  $(x, t, y, \tau) \mapsto g(y, \tau, u_0(x, t))$  lies in  $L^2(Q_T; \mathcal{C}_{\text{per}}(Y \times Z))$ , so that we have  $g^\varepsilon(u_0) \rightarrow g(\cdot, \cdot, u_0)$  in  $L^2(Q_T)$ -2s as  $\varepsilon \rightarrow 0$ . Now, for  $f \in L^2(Q_T; \mathcal{C}_{\text{per}}(Y \times Z))$ ,

$$\begin{aligned} & \int_{Q_T} g^\varepsilon(u_\varepsilon) f^\varepsilon dxdt - \iint_{Q_T \times Y \times Z} g(\cdot, \cdot, u_0) f dxdt dyd\tau \\ &= \int_{Q_T} (g^\varepsilon(u_\varepsilon) - g^\varepsilon(u_0)) f^\varepsilon dxdt + \int_{Q_T} g^\varepsilon(u_0) f^\varepsilon dxdt \\ & \quad - \iint_{Q_T \times Y \times Z} g(\cdot, \cdot, u_0) f dxdt dyd\tau. \end{aligned}$$

Using the inequality

$$\left| \int_{Q_T} (g^\varepsilon(u_\varepsilon) - g^\varepsilon(u_0)) f^\varepsilon dxdt \right| \leq C \|u_\varepsilon - u_0\|_{L^2(Q_T)} \|f^\varepsilon\|_{L^2(Q_T)}$$

in conjunction with the above convergence results we get readily the result.  $\square$

**Remark 2.** From the Lipschitz property of the function  $g$  above we may get more information on the limit of the sequence  $g^\varepsilon(u_\varepsilon)$ . Indeed, since  $|g^\varepsilon(u_\varepsilon) - g^\varepsilon(u_0)| \leq C|u_\varepsilon - u_0|$ , we deduce the following convergence result:

$$g^\varepsilon(u_\varepsilon) \rightarrow \tilde{g}(u_0) \text{ in } L^2(Q_T) \text{ as } \varepsilon \rightarrow 0$$

where  $\tilde{g}(u_0)(x, t) = \int_{Y \times Z} g(y, \tau, u_0(x, t)) dyd\tau$ .

We will need the following spaces:

$$\mathbb{F}_0^{1,p} = L^p(0, T; W_0^{1,p}(Q)) \times L^p(Q_T; \mathcal{X})$$

(where  $\mathcal{X}$  is either  $\mathcal{V}$  or  $\mathcal{W}$ ) and

$$\mathcal{F}_0^\infty = \mathcal{C}_0^\infty(Q_T) \times [\mathcal{C}_0^\infty(Q_T) \otimes \mathcal{E}]$$

where we recall that  $\mathcal{W} = \{v \in \mathcal{V} : \rho \partial v / \partial \tau \in \mathcal{V}'\}$  with  $\mathcal{V} = L_{\text{per}}^p(Z; W_{\# \rho}^{1,p}(Y))$ , and  $\mathcal{E} = \mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\# \rho}(Y)]$ .  $\mathbb{F}_0^{1,p}$  is a Banach space under the norm

$$\|(u_0, u_1)\|_{\mathbb{F}_0^{1,p}} = \|u_0\|_{L^p((0,T); W_0^{1,p}(Q))} + \|u_1\|_{L^p(Q_T; \mathcal{X})}$$

with the further property that  $\mathcal{F}_0^\infty$  is dense in  $\mathbb{F}_0^{1,p}$ ; this obviously follows from Lemma 2.

## 5. HOMOGENIZATION RESULTS

**5.1. Global homogenized problem.** For a function  $u \in L^p(0, T; W_0^{1,p}(Q))$ , we shall denote by  $u'$  the partial derivative  $\partial u / \partial t$  defined in a distributional sense on  $\mathcal{D}'(Q_T)$ . Let  $E$  be an ordinary sequence of positive real numbers  $\varepsilon$  converging to 0 with  $\varepsilon$ . We assume throughout this section that  $p \geq 2$ . By the strong relative compactness of the family  $(u_\varepsilon)_{\varepsilon > 0}$  (see Proposition 1), there exist a subsequence  $E'$  from  $E$  and a function  $u_0 \in L^p(0, T; W_0^{1,p}(Q))$  such that, as  $E' \ni \varepsilon \rightarrow 0$ ,

$$u_\varepsilon \rightarrow u_0 \text{ in } L^2(Q_T). \quad (5.1)$$

Let  $u_1 \in L^p(Q_T \times Z; W_{\# \rho}^{1,p}(Y))$  be the function determined by the Theorem 4 such that, as  $E' \ni \varepsilon \rightarrow 0$ ,

$$\frac{\partial u_\varepsilon}{\partial x_j} \rightarrow \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \text{ in } L^p(Q_T)\text{-2s } (1 \leq j \leq N). \quad (5.2)$$

The first important result of this section is the following

**Theorem 5.** *The couple  $(u_0, u_1)$  determined above by (5.1)-(5.2) solves the following variational problem:*

$$\left\{ \begin{array}{l} (u_0, u_1) \in L^p(0, T; W_0^{1,p}(Q)) \times L^p(Q_T; \mathcal{V}) : \\ \int_0^T (u'_0, v_0) dxdt + \iint_{Q_T \times Y \times Z} a(\cdot, Du_0 + D_y u_1) \cdot (Dv_0 + D_y v_1) dxdt dyd\tau \\ = \iint_{Q_T \times Y \times Z} g(y, \tau, u_0) v_1 dxdt dyd\tau - \iint_{Q_T \times Y \times Z} G(y, \tau, u_0) \cdot Dv_0 dxdt dyd\tau \\ - \iint_{Q_T \times Y \times Z} (\partial_r G(y, \tau, u_0) \cdot (Du_0 + D_y u_1)) v_0 dxdt dyd\tau \\ \text{for all } (v_0, v_1) \in \mathcal{F}_0^\infty, \text{ if } 0 < k < 2; \end{array} \right. \quad (5.3)$$

$$\left\{ \begin{array}{l} (u_0, u_1) \in L^p(0, T; W_0^{1,p}(Q)) \times L^p(Q_T; \mathcal{W}) : \\ \int_0^T (u'_0, v_0) dxdt + \int_{Q_T} [\rho \frac{\partial u_1}{\partial \tau}, v_1] dxdt \\ = - \iint_{Q_T \times Y \times Z} a(\cdot, Du_0 + D_y u_1) \cdot (Dv_0 + D_y v_1) dxdt dyd\tau \\ + \iint_{Q_T \times Y \times Z} g(y, \tau, u_0) v_1 dxdt dyd\tau - \iint_{Q_T \times Y \times Z} G(y, \tau, u_0) \cdot Dv_0 dxdt dyd\tau \\ - \iint_{Q_T \times Y \times Z} (\partial_r G(y, \tau, u_0) \cdot (Du_0 + D_y u_1)) v_0 dxdt dyd\tau \\ \text{for all } (v_0, v_1) \in \mathcal{F}_0^\infty, \text{ if } k = 2; \end{array} \right. \quad (5.4)$$

and

$$\left\{ \begin{array}{l} (u_0, u_1) \in L^p(0, T; W_0^{1,p}(Q)) \times L^p(Q_T; W_{\# \rho}^{1,p}(Y)) : \\ \int_0^T (u'_0, v_0) dxdt + \iint_{Q_T \times Y} \bar{a}(\cdot, Du_0 + D_y u_1) \cdot (Dv_0 + D_y v_1) dxdt dy \\ = \iint_{Q_T \times Y \times Z} \bar{g}(y, u_0) v_1 dxdt dy - \iint_{Q_T \times Y} \bar{G}(y, u_0) \cdot Dv_0 dxdt dy \\ - \iint_{Q_T \times Y} (\bar{\partial}_r \bar{G}(y, u_0) \cdot (Du_0 + D_y u_1)) v_0 dxdt dy \\ \text{for all } (v_0, v_1) \in \mathcal{C}_0^\infty(Q_T) \times (\mathcal{C}_0^\infty(Q_T) \otimes \mathcal{D}_{\# \rho}(Y)), \text{ if } k > 2, \end{array} \right. \quad (5.5)$$

where  $\bar{a}(\cdot, Du_0 + D_y u_1) = \int_0^1 a(\cdot, Du_0 + D_y u_1) d\tau$ ,  $\bar{g}(y, u_0) = \int_0^1 g(y, \tau, u_0) d\tau$  (and a similar definition for  $\bar{G}(y, u_0)$  and  $\bar{\partial}_r \bar{G}(y, u_0)$ ).

*Proof.* The proof will be done in three steps, according to the values of the parameter  $k$ .

**Step 1: Case where  $0 < k < 2$ .** Let  $\Phi = (\psi_0, \psi_1) \in \mathcal{F}_0^\infty$ , and define

$$\Phi_\varepsilon(x, t) = \psi_0(x, t) + \varepsilon \psi_1 \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k} \right), \quad (x, t) \in Q_T.$$

We recall that  $\psi_0 \in \mathcal{C}_0^\infty(Q_T)$  and  $\psi_1 \in \mathcal{C}_0^\infty(Q_T) \otimes (\mathcal{D}_{\text{per}}(Z) \otimes [\mathcal{D}_{\# \rho}(Y)])$ . Then  $\Phi_\varepsilon \in \mathcal{C}_0^\infty(Q_T)$  and, using it as a test function in the variational formulation of (1.1), we get

$$- \int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt + \int_{Q_T} a^\varepsilon(\cdot, Du_\varepsilon) \cdot D\Phi_\varepsilon dxdt - \frac{1}{\varepsilon} \int_{Q_T} g^\varepsilon(u_\varepsilon) \Phi_\varepsilon dxdt = 0. \quad (5.6)$$

We consider the terms in (5.6) respectively.

As regards the first term on the left-hand side of (5.6), we have

$$\begin{aligned} \int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt &= \int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \psi_0}{\partial t} dxdt + \varepsilon \int_{Q_T} \rho^\varepsilon u_\varepsilon \left( \frac{\partial \psi_1}{\partial t} \right)^\varepsilon dxdt \\ &\quad + \varepsilon^{2-k} \int_{Q_T} \frac{1}{\varepsilon} \rho^\varepsilon u_\varepsilon \left( \frac{\partial \psi_1}{\partial \tau} \right)^\varepsilon dxdt. \end{aligned}$$

In view of Lemma 4, the integral  $\int_{Q_T} \frac{1}{\varepsilon} \rho^\varepsilon u_\varepsilon \left( \frac{\partial \psi_1}{\partial \tau} \right)^\varepsilon dxdt$  converges (recall that  $M_y(\rho \frac{\partial \psi_1}{\partial \tau}) = 0$ ). On the other hand, since  $\rho^\varepsilon \rightarrow \int_Y \rho dy = 1$  in  $L^2(Q)$ -weak and  $u_\varepsilon \rightarrow u_0$  in  $L^2(Q_T)$ , it is immediate that

$$\int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \psi_0}{\partial t} dxdt \rightarrow \int_{Q_T} u_0 \frac{\partial \psi_0}{\partial t} dxdt \text{ as } \varepsilon \rightarrow 0.$$

We are led to

$$\int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt \rightarrow - \int_0^T (u'_0(t), \psi_0(\cdot, t)) dt.$$

As far as the second term on the left-hand side of (5.6) is concerned, due to the monotonicity of the function  $a(x, t, y, \tau, \cdot)$  we can argue as in [21] (see also [20]) to get

$$\int_{Q_T} a^\varepsilon(\cdot, Du_\varepsilon) \cdot D\Phi_\varepsilon dxdt \rightarrow \iint_{Q_T \times Y \times Z} a(\cdot, Du_0 + D_y u_1) \cdot (D\psi_0 + D_y \psi_1) dxdt dy d\tau.$$

Finally, for the last term, we have

$$\frac{1}{\varepsilon} \int_{Q_T} g^\varepsilon(u_\varepsilon) \Phi_\varepsilon dxdt = \frac{1}{\varepsilon} \int_{Q_T} g^\varepsilon(u_\varepsilon) \psi_0 dxdt + \int_{Q_T} g^\varepsilon(u_\varepsilon) \psi_1^\varepsilon dxdt.$$

It is immediate that

$$\int_{Q_T} g^\varepsilon(u_\varepsilon) \psi_1^\varepsilon dxdt \rightarrow \iint_{Q_T \times Y \times Z} g(u_0) \psi_1 dxdt dy d\tau.$$

For  $\frac{1}{\varepsilon} \int_{Q_T} g^\varepsilon(u_\varepsilon) \psi_0 dxdt$ , we use the decomposition

$$\frac{1}{\varepsilon} g^\varepsilon(u_\varepsilon) = \operatorname{div} G^\varepsilon(u_\varepsilon) - \partial_r G^\varepsilon(u_\varepsilon) \cdot Du_\varepsilon$$

to get

$$\begin{aligned} \frac{1}{\varepsilon} \int_{Q_T} g^\varepsilon(u_\varepsilon) \Phi_\varepsilon dxdt &= - \int_{Q_T} G^\varepsilon(u_\varepsilon) \cdot D\psi_0 dxdt - \int_{Q_T} (\partial_r G^\varepsilon(u_\varepsilon) \cdot Du_\varepsilon) \psi_0 dxdt \\ &= I_1 + I_2. \end{aligned}$$

We infer from Lemma 5 that

$$I_1 \rightarrow - \iint_{Q_T \times Y \times Z} G(u_0) \cdot D\psi_0 dxdt dy d\tau.$$

Since the function  $\partial_r G$  is Lipschitz continuous with respect to  $r$  and periodic with respect to  $y, \tau$ , the use of Remark 2 yields

$$I_2 \rightarrow - \iint_{Q_T \times Y \times Z} (\partial_r G(u_0) \cdot (Du_0 + D_y u_1)) \psi_0 dxdt dy d\tau;$$

indeed, this can be verified by using the definition of the strong two-scale convergence [1, 23], noting that in our case, the sequence  $\partial_r G^\varepsilon(u_\varepsilon)$  strongly two-scale converges towards  $\partial_r G(u_0)$ .

Putting together all the above facts we are led to (5.3).

*Step 2: Case where  $k = 2$ .* In this case the procedure is the same as in the previous one. Thus, as it can be seen from the proof of the case  $0 < k < 2$ , we will only deal with the term  $\int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt$ . However, another important fact is to check that  $u_1$  belongs to  $L^p(Q_T; \mathcal{W})$ . This last part will be accomplished in the

next subsection. Returning to (5.6) and considering the first term there, we pass to the limit in the equality

$$\int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt = \int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \psi_0}{\partial t} dxdt + \int_{Q_T} \varepsilon \rho^\varepsilon u_\varepsilon \frac{\partial \psi_1^\varepsilon}{\partial t} dxdt$$

using Corollary 1 and we get

$$\lim_{E' \ni \varepsilon \rightarrow 0} \int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt = - \int_0^T (u'_0(t), \psi_0(\cdot, t)) dt - \int_{Q_T} \left[ \rho \frac{\partial u_1}{\partial \tau}, \psi_1 \right] dxdt,$$

and we hence derive (5.4).

*Step 3: Case where  $k > 2$ .* As in the preceding step, we only need to compute the limit (as  $E' \ni \varepsilon \rightarrow 0$ ) of the term  $\int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt$ . Before we can do this, we must first show that the corrector term  $u_1$  does not depend on  $\tau$ . This will allow us to take the test functions independent of  $\tau$ , that is,  $\psi_1 \in \mathcal{C}_0^\infty(Q_T) \otimes [\mathcal{D}_{\# \rho}(Y)]$ , i.e.,  $\Phi_\varepsilon(x, t) = \psi_0(x, t) + \varepsilon \psi_1(x, t, x/\varepsilon)$ . This will therefore lead at once to

$$\int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdt \rightarrow - \int_0^T (u'_0(t), \psi_0(\cdot, t)) dt \text{ as } E' \ni \varepsilon \rightarrow 0.$$

So, let

$$\psi_\varepsilon(x, t) = \varepsilon^{k-1} \psi \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^k} \right), (x, t) \in Q_T,$$

where  $\psi(x, t, y, \tau) = \varphi(x, t) \theta(y) \chi(\tau)$  with  $\varphi \in \mathcal{C}_0^\infty(Q_T)$ ,  $\theta \in \mathcal{D}_{\# \rho}(Y)$  and  $\chi \in \mathcal{D}_{\text{per}}(Z)$ . Then  $\psi_\varepsilon \in \mathcal{C}_0^\infty(Q_T)$  and as in (5.6) we have

$$- \int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \psi_\varepsilon}{\partial t} dxdt + \int_{Q_T} a^\varepsilon(\cdot, Du_\varepsilon) \cdot D\psi_\varepsilon dxdt - \frac{1}{\varepsilon} \int_{Q_T} g^\varepsilon(u_\varepsilon) \psi_\varepsilon dxdt = 0. \quad (5.7)$$

Because  $k > 2$ , the last two terms in the left-hand side of (5.7) go to zero as  $E' \ni \varepsilon \rightarrow 0$ . For the first one we have

$$\int_{Q_T} \rho^\varepsilon u_\varepsilon \frac{\partial \psi_\varepsilon}{\partial t} dxdt = \varepsilon^k \int_{Q_T} \frac{1}{\varepsilon} \rho^\varepsilon u_\varepsilon \left( \frac{\partial \psi}{\partial t} \right)^\varepsilon dxdt + \int_{Q_T} \frac{1}{\varepsilon} \rho^\varepsilon u_\varepsilon \left( \frac{\partial \psi}{\partial \tau} \right)^\varepsilon dxdt.$$

Passing to the limit in the above equation using Lemma 4 gives in (5.7)

$$\iint_{Q_T \times Y \times Z} \rho u_1 \frac{\partial \psi}{\partial \tau} dxdt dy d\tau = 0,$$

and using the arbitrariness of  $\varphi$  and  $\theta$ , we get

$$\int_0^1 \rho(y) u_1(x, t, y, \tau) \frac{\partial \chi}{\partial \tau}(\tau) d\tau = 0,$$

which is equivalent to  $u_1$  is independent of  $\tau$ . This ends the proof of Step 3. We are partially done (since we need to check that  $u_1$ , in the case  $k = 2$ , lies in  $L^p(Q_T; \mathcal{W})$ ).  $\square$

**5.2. Homogenized equation.** In this section we consider each of the Eq. (5.3)-(5.5) separately. Let us first and foremost deal with (5.3).

Equation (5.3) is equivalent to the following system:

$$\begin{cases} \iint_{Q_T \times Y \times Z} a(\cdot, Du_0 + D_y u_1) \cdot D_y v_1 dx dt dy d\tau \\ = \iint_{Q_T \times Y \times Z} g(y, \tau, u_0) v_1 dx dt dy d\tau \text{ for all } v_1 \in \mathcal{C}_0^\infty(Q_T) \otimes \mathcal{E} \end{cases} \quad (5.8)$$

and

$$\begin{cases} \int_0^T (u'_0, v_0) dx dt + \iint_{Q_T \times Y \times Z} a(\cdot, Du_0 + D_y u_1) \cdot Dv_0 dx dt dy d\tau \\ + \iint_{Q_T \times Y \times Z} G(y, \tau, u_0) \cdot Dv_0 dx dt dy d\tau \\ + \iint_{Q_T \times Y \times Z} (\partial_r G(y, \tau, u_0) \cdot (Du_0 + D_y u_1)) v_0 dx dt dy d\tau = 0 \\ \text{for all } v_0 \in \mathcal{C}_0^\infty(Q_T). \end{cases} \quad (5.9)$$

As far as (5.8) is concerned, let  $(x, t) \in Q_T$  and let  $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$  be freely fixed. Let  $\pi(x, t, r, \xi)$  be defined by the so-called cell problem

$$\begin{cases} \pi(x, t, r, \xi) \in \mathcal{V} = L_{\text{per}}^p(Z; W_{\# \rho}^{1,p}(Y)) : \\ \int_{Y \times Z} a(\cdot, \xi + D_y \pi(x, t, r, \xi)) \cdot D_y w dy d\tau = \int_{Y \times Z} g(y, \tau, r) w dy d\tau \\ \text{for all } w \in \mathcal{V}. \end{cases} \quad (5.10)$$

Since  $g(y, \tau, r) = \text{div}_y G(y, \tau, r)$ , we have

$$\int_{Y \times Z} g(y, \tau, r) w dy d\tau = - \int_{Y \times Z} G(y, \tau, r) \cdot D_y w dy d\tau,$$

from which we deduce that the right-hand side of (5.10) is a continuous linear functional on  $\mathcal{V}$ . It therefore follows from classical results that Eq. (5.10) admits at least a solution. Moreover if  $\pi_1 \equiv \pi_1(x, t, r, \xi)$  and  $\pi_2 \equiv \pi_2(x, t, r, \xi)$  are two solutions of (5.10), then we must have

$$\int_{Y \times Z} (a(\cdot, r, \xi + D_y \pi_1) - a(\cdot, r, \xi + D_y \pi_2)) \cdot (D_y \pi_1 - D_y \pi_2) dy d\tau = 0,$$

and so, by [part (i) of] (3.3),  $D_y \pi_1 = D_y \pi_2$ , which means that  $\pi_1 - \pi_2$  is a constant function of  $y$ . But then by the condition  $M_y(\rho \pi_1) = M_y(\rho \pi_2) = 0$  (recall that  $\pi_1$  and  $\pi_2$  are in  $\mathcal{V} = L_{\text{per}}^p(Z; W_{\# \rho}^{1,p}(Y))$ ) we deduce that  $\pi_1 = \pi_2$ . Next, taking in particular  $r = u_0(x, t)$  and  $\xi = Du_0(x, t)$  with  $(x, t)$  arbitrarily chosen in  $Q_T$ , and then choosing in (5.8) the particular test functions  $v_1(x, t) = \varphi(x, t)w$  ( $(x, t) \in Q_T$ ) with  $\varphi \in \mathcal{C}_0^\infty(Q_T)$  and  $w \in \mathcal{E}$ , and finally comparing the resulting equation with (5.10) (note that  $\mathcal{E}$  is dense in  $\mathcal{V}$ ), the uniqueness of the solution to (5.10) tells us that  $u_1 = \pi(\cdot, u_0, Du_0)$ , where the right-hand side of the preceding equality stands for the function  $(x, t) \mapsto \pi(x, t, u_0(x, t), Du_0(x, t))$  from  $Q_T$  into  $\mathcal{V}$ .

We have just proved the

**Proposition 3.** *The solution of the variational problem (5.8) is unique.*

Let us now deal with the variational problem (5.9). For that, set

$$q(x, t, r, \xi) = \int_{Y \times Z} a(x, t, \cdot, \cdot, \xi + D_y \pi(x, t, r, \xi)) dy d\tau$$

and

$$q_0(x, t, r, \xi) = \int_{Y \times Z} \partial_r g(y, \tau, r) \pi(x, t, r, \xi) dy d\tau$$

for  $(x, t) \in Q_T$  and  $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$  arbitrarily fixed. With this in mind, we have following

**Proposition 4.** *The solution  $u_0$  to the variational problem (5.9) solves the following boundary value problem:*

$$\begin{cases} \frac{\partial u_0}{\partial t} = \operatorname{div} q(\cdot, \cdot, u_0, Du_0) + q_0(\cdot, \cdot, u_0, Du_0) & \text{in } Q_T \\ u_0 = 0 & \text{on } \partial Q \times (0, T) \\ u_0(x, 0) = u^0(x) & \text{in } Q. \end{cases} \quad (5.11)$$

Moreover any subsequential limit in  $L^2(Q_T)$  of the sequence  $(u_\varepsilon)_{\varepsilon>0}$  is solution to (5.11).

*Proof.* Substituting  $u_1 = \pi(\cdot, u_0, Du_0)$  in (5.9) and using the obvious equalities

$$\begin{aligned} - \iint_{Q_T \times Y \times Z} G(y, \tau, u_0) \cdot Dv_0 dx dt dy d\tau &= \iint_{Q_T \times Y \times Z} (\partial_r G(y, \tau, u_0) \cdot Du_0) v_0 dx dt dy d\tau, \\ - \iint_{Q_T \times Y \times Z} (\partial_r G(y, \tau, u_0) \cdot D_y u_1) v_0 dx dt dy d\tau &= \iint_{Q_T \times Y \times Z} \partial_r g(y, \tau, u_0) u_1 v_0 dx dt dy d\tau, \end{aligned}$$

Eq. (5.9) becomes

$$\begin{cases} \int_0^T (u'_0, v_0) dx dt + \iint_{Q_T \times Y \times Z} a(\cdot, Du_0 + D_y \pi(\cdot, u_0, Du_0)) \cdot Dv_0 dx dt dy d\tau \\ = \iint_{Q_T \times Y \times Z} \partial_r g(y, \tau, u_0) \pi(\cdot, u_0, Du_0) v_0 dx dt dy d\tau \end{cases} \text{ for all } v_0 \in \mathcal{C}_0^\infty(Q_T),$$

which is nothing else, but the variational formulation of (5.11).  $\square$

To conclude the study in the case when  $0 < k < 2$ , we have the following

**Theorem 6.** *Let  $2 \leq p < \infty$ . Assume hypotheses **A1-A5** hold. For each  $\varepsilon > 0$  let  $u_\varepsilon$  be the unique solution to (1.1). Then there exists a subsequence of  $\varepsilon$  not relabeled such that  $u_\varepsilon \rightarrow u_0$  in  $L^2(Q_T)$  where  $u_0 \in L^p(0, T; W_0^{1,p}(Q))$  is solution to (5.11).*

The case when  $k > 2$  is quite similar to that when  $0 < k < 2$ . Now, let us consider the case where  $k = 2$ . In that case, all we need to check is that the solution  $u_1$  of the microscopic problem is unique and belongs to  $L^p(Q_T; \mathcal{W})$  as announced in Theorem 5. For that purpose, we begin by checking that  $u_1$  is the solution to the following variational problem:

$$\begin{cases} \int_{Q_T} [\rho \frac{\partial u_1}{\partial \tau}, v_1] dx dt + \iint_{Q_T \times Y \times Z} a(\cdot, Du_0 + D_y u_1) \cdot D_y v_1 dx dt dy d\tau \\ = \iint_{Q_T \times Y \times Z} g(y, \tau, u_0) v_1 dx dt dy d\tau \end{cases} \text{ for all } v_1 \in \mathcal{C}_0^\infty(Q_T) \otimes \mathcal{E}. \quad (5.12)$$

Fix  $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$  and  $(x, t) \in Q_T$ , and consider the cell problem

$$\begin{cases} \pi \equiv \pi(x, t, r, \xi) \in \mathcal{V} = L_{\text{per}}^p(Z; W_{\# \rho}^{1,p}(Y)) \\ [\rho \frac{\partial \pi}{\partial \tau}, w] + \int_{Y \times Z} a(\cdot, \xi + D_y \pi) \cdot D_y w dy d\tau = \int_{Y \times Z} g(y, \tau, r) w dy d\tau \\ \text{for all } w \in \mathcal{E} = \mathcal{D}_{\text{per}}(Z) \otimes \mathcal{D}_{\# \rho}(Y). \end{cases} \quad (5.13)$$

Assume for a while that the solution of (5.13) exists. Then, for the same reasons as in the case where  $0 < k < 2$ , the linear functional  $L : w \mapsto \int_{Y \times Z} g(y, \tau, r) w dy d\tau$  defined on  $\mathcal{V}$  verifies the property: there is a positive constant  $c$  independent of  $w$  such that

$$|L(w)| \leq c \|w\|_{\mathcal{V}} \text{ for all } w \in \mathcal{V}.$$

Likewise there exists another constant  $k > 0$  such that

$$\left| \int_{Y \times Z} a(\cdot, \xi + D_y \pi) \cdot D_y w dy d\tau \right| \leq k \|w\|_{\mathcal{V}}.$$



We deduce from the above facts that the linear functional  $w \mapsto [\rho \frac{\partial \pi}{\partial \tau}, w]$ , defined on  $\mathcal{E}$ , is continuous when endowing  $\mathcal{E}$  with the  $\mathcal{V}$ -norm. From the density of  $\mathcal{E}$  in  $\mathcal{V}$  we get readily  $\rho \frac{\partial \pi}{\partial \tau} \in \mathcal{V}'$ , so that  $\pi$  lies in  $\mathcal{W}$ . Since  $\mathcal{E}$  in  $\mathcal{W}$ , Eq. (5.13) still holds for  $w \in \mathcal{W}$ . Therefore by equality (4.2) (in Proposition 2) we deduce that  $[\rho \frac{\partial \pi}{\partial \tau}, \pi] = 0$ . The uniqueness of the solution of (5.13) follows from that. So it remains to show that Eq. (5.13) possesses at least a solution. But this equation is the variational formulation of the following equation:

$$\begin{cases} \pi \in \mathcal{W} : \\ \rho \frac{\partial \pi}{\partial \tau} = \operatorname{div}_y a(\cdot, \xi + D_y \pi) + g(\cdot, \cdot, r). \end{cases} \quad (5.14)$$

In view of the properties of the operator  $\mathcal{R}$  defined in Section 4, we see immediately by [16] (see also [3]) that the above equation admits at least a solution. Now, taking  $r = u_0(x, t)$  and  $\xi = Du_0(x, t)$  and arguing as in the case where  $0 < k < 2$ , we obtain  $u_1 = \pi(\cdot, \cdot, u_0, Du_0)$ . By the preceding equality, we have shown, as claimed, that  $u_1$  lies in  $L^p(Q_T; \mathcal{W})$ , thereby concluding the proof of Theorem 5. This also shows that even in this case, the homogenized equation still has the form (5.11).

**5.3. Some uniqueness results and convergence of the sequence  $(u_\varepsilon)_{\varepsilon > 0}$ .** In order to find a uniqueness result for the solution of the problem (5.11), we need to know the properties of the homogenized coefficients. The properties of the function  $q$  are classically known. However it is difficult to have the general properties of the function  $q_0$ . But we will nevertheless show that in some cases, there is uniqueness. For this, we will restrict the study to a special case: we assume that the function  $\lambda \mapsto a(x, t, y, \tau, \lambda)$ , from  $\mathbb{R}^N$  into itself is linear, that is, there exists a family  $\{a_{ij}\}_{1 \leq i, j \leq N} \subset \mathcal{C}(\overline{Q}_T; L^\infty(\mathbb{R}_y^{N+1}))$  (thanks to (3.1) and parts (ii) and (iii) of (3.3)), such that

$$a_i(x, t, y, \tau, \lambda) = \sum_{j=1}^N a_{ij}(x, t, y, \tau) \lambda_j \text{ for all } \lambda \in \mathbb{R}^N \text{ (} 1 \leq i \leq N \text{)}.$$

In the sequel, we assume  $p = 2$ . It is clear that the results obtained in the preceding sections are still valid in this case. From the periodicity assumption on  $a(x, t, \cdot, \cdot, \lambda)$ , it is clear that the functions  $a_{ij}(x, t, \cdot, \cdot)$  are  $Y \times Z$ -periodic.

Set  $b = (a_{ij})_{1 \leq i, j \leq N}$  (the matrix derived from the coefficients  $a_{ij}$ ) and let us focused our attention on the special case where  $k = 2$ , which seems to be the more involved. The cell problem (5.14) takes the form

$$\begin{cases} \pi \equiv \pi(x, t, r, \xi) \in \mathcal{W} : \\ \rho \frac{\partial \pi}{\partial \tau} = \operatorname{div}_y (b(x, t, \cdot, \cdot)(\xi + D_y \pi)) + g(\cdot, \cdot, r). \end{cases} \quad (5.15)$$

We know that this equation has a unique solution. But it can be easily seen that the solution of the above equation expresses under the form

$$\pi(x, t, r, \xi)(y, \tau) = \chi(x, t, y, \tau) \cdot \xi + w_1(x, t, y, \tau, r) \quad (5.16)$$

where  $\chi$  and  $w_1$  are respective unique solutions to the following equations

$$\rho \frac{\partial \chi}{\partial \tau} - \operatorname{div}_y (b(x, t) D_y \chi) = \operatorname{div}_y b \text{ in } \mathcal{W}', \quad \chi = \chi(x, t, \cdot, \cdot) \in (\mathcal{W})^N,$$

and

$$\rho \frac{\partial w_1}{\partial \tau} - \operatorname{div}_y (b(x, t) D_y w_1) = g(\cdot, \cdot, r) \text{ in } \mathcal{W}', \quad w_1 = w_1(x, t, \cdot, \cdot, r) \in \mathcal{W}$$

where  $b(x, t)$  stands for the matrix  $(a_{ij}(x, t, \cdot, \cdot))_{1 \leq i, j \leq N}$ . The existence and uniqueness of  $\chi$  and  $w_1$  is ensured by a classical result [16].

Now, taking  $r = u_0(x, t)$  and  $\xi = Du_0(x, t)$  in (5.15), it follows from (5.16) that

$$u_1(x, t, y, \tau) = \chi(x, t, y, \tau) \cdot Du_0(x, t) + w_1(x, t, y, \tau, u_0(x, t)).$$

Now, going back to the variational formulation of (5.4) with the function  $u_1$  replaced by the above expression, we end up with

$$\begin{cases} \int_0^T (u'_0, v_0) dxdt + \int_{Q_T} (\widehat{b}(x, t) Du_0) \cdot Dv_0 dxdt \\ + \iint_{Q_T \times Y \times Z} b(x, t) D_y w_1(x, t, u_0) \cdot Dv_0 dxdt dyd\tau \\ = \iint_{Q_T \times Y \times Z} \partial_r g(y, \tau, u_0) (\chi(x, t, y, \tau) \cdot Du_0(x, t)) dxdt dyd\tau \\ + \iint_{Q_T \times Y \times Z} \partial_r g(y, \tau, u_0) w_1(x, t, y, \tau, u_0(x, t)) v_0 dxdt dyd\tau \end{cases} \text{ for all } v_0 \in C_0^\infty(Q_T),$$

where  $\widehat{b}(x, t) = \int_{Y \times Z} b(x, t) (I + D_y \chi) dyd\tau$  is the homogenized matrix,  $I$  being denoting the unit  $N \times N$  matrix. Setting

$$\begin{aligned} F_1(x, t, r) &= \int_{Y \times Z} b(x, t) D_y w_1(x, t, y, \tau, r) dyd\tau; \\ F_2(x, t, r) &= \int_{Y \times Z} \partial_r g(y, \tau, r) \chi(x, t, y, \tau) dyd\tau; \\ F_3(x, t, r) &= \int_{Y \times Z} \partial_r g(y, \tau, r) w_1(x, t, y, \tau, r) dyd\tau, \end{aligned}$$

we are led to the following result.

**Proposition 5.** *The solution  $u_0$  to the variational problem (5.4) solves the following boundary value problem:*

$$\begin{cases} \frac{\partial u_0}{\partial t} = \operatorname{div} (\widehat{b}(x, t) Du_0) + \operatorname{div} F_1(x, t, u_0) - F_2(x, t, u_0) \cdot Du_0 - F_3(x, t, u_0) & \text{in } Q_T \\ u_0 = 0 & \text{on } \partial Q \times (0, T) \\ u_0(x, 0) = u^0(x) & \text{in } Q. \end{cases} \quad (5.17)$$

As in [2], it can be checked straightforwardly that the functions  $F_i(x, t, \cdot)$  ( $1 \leq i \leq 3$ ) are Lipschitz continuous functions. This therefore ensures the uniqueness of the solution to (5.17), and the following result holds true.

**Theorem 7.** *Assume hypotheses **A1-A5** hold with  $p = 2$ . For each  $\varepsilon > 0$  let  $u_\varepsilon$  be the unique solution to (1.1). Then  $u_\varepsilon \rightarrow u_0$  in  $L^2(Q_T)$  as  $\varepsilon \rightarrow 0$ , where  $u_0 \in L^2(0, T; H_0^1(Q))$  is the unique solution to (5.17).*

*Proof.* By the uniqueness of the solution to (5.17), the result follows in an obvious way.  $\square$

The same remark as above holds in all the other cases (as far as the parameter  $k$  is concerned), so that we are justified in saying that Theorem 7 holds for any positive value of the parameter  $k$ . This shows the convergence of the sequence  $(u_\varepsilon)_{\varepsilon > 0}$  when the function  $a(x, t, y, \tau, \lambda)$  is linear with respect to  $\lambda$ . Also we recover the results by Allaire and Piatnitski [2] (when setting in our situation  $k = 2$ ) when the function  $a(x, t, y, \tau, \lambda)$  is linear and does not depend on the variable  $x, t$ . We can therefore argue that our work generalize the one of the previous authors.

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